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Combustion in plane steady compressible flow: general considerations and gasdynamical adjustment regions

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By specializing to the case of unit Lewis number and Prandtl number equal to $\frac{3}{4}$, a number of general results for the structure of a plane steady compressible flow field, within which chemical energy is being liberated by a simple Arrhenius type of combustion reaction, can be acquired by the use of elementary arguments. The field is of the semi-infinite variety, with a 'flameholder' presumed to exist at the origin of coordinates. In these circumstances it is necessary to specify the velocity gradient at inlet to the system or, equivalently, the pressure difference across the field. These quantities cannot be selected arbitrarily, and the nature and extent of the restrictions upon them is fully explored. Since the Mach number of the stream is hypothesized to be a quantity of order unity, local Damköhler numbers are always small. Therefore the field is shown to consist of relatively long regions within which the combustion activity takes place, with embedded thin domains of rapid, almost chemically inert, gasdynamical adjustment, whose dimension is typically that of the conventional shock wave. When the inlet Mach number is less than unity the gasdynamical adjustment domains are always adjacent to the origin, and this is also true under most supersonic inlet conditions.

However, there are some special circumstances for which the shock is detached from the flameholder and is established in the middle of the combustion activity. A specific example is provided by a shock within the induction domain.

These special circumstances are shown to be ultrasensitive to pressure difference across the whole domain. It is also shown that wholly supersonic combustion does exist, but only under similar conditions of extreme sensitivity to pressure difference.

The general arguments are supported and illuminated by asymptotic analysis based on the large activation energy of the Arrhenius reaction. Space precludes a full asymptotic treatment of the combustion activity but a companion paper that analyses these parts of the general field is being prepared in collaboration with D. R. Kassoy. Analysis of the shock within the induction domain, together with results from the case of subsonic inlet Mach numbers, shows that gasdynamical effects can prevent ignition by channelling combustion energy into kinetic energy of the flow at the expense of thermal energy.

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1. Introduction

When the Prandtl and Lewis numbers have the constant values of $\frac{3}{4}$ and 1 respectively, a satisfactory description of a compressible flow field that sustains a simple combustion reaction is provided by the pair of differential equations†

$$T_s' - T_s'' = \{A_4 \theta^s M^{-2} e^{-\theta/T}\}(T_{sb} - T_s) \equiv R, \tag{1}$$

$$u' - \sigma_0 = u - 1 + \frac{1}{\gamma M^2}\left(\frac{T}{u} - 1\right), \tag{2}$$

where

$$T_s \equiv T + \tfrac{1}{2}(\gamma - 1) M^2 u^2 \tag{3}$$

is the dimensionless stagnation temperature and σ_0 is the value of u' at $\xi = 0$. The absolute temperature T and gas velocity u are measured in units of the inlet temperature and gas velocity at the 'flameholder' that is presumed to exist at the location $\xi = 0$. M is the Mach number of the inlet flow, γ is the (constant) ratio of the frozen specific heats and θ is the dimensionless activation energy of the combustion reaction; T_{sb} is the final or 'burnt' value of the stagnation temperature, that is achieved as $\xi \to \infty$, and A_4 is a pre-exponential number, whose value is essentially around unity; s is a small (i.e. 1 to 2, say) index; a prime, e.g. u', indicates a differentiation with respect to the coordinate ξ; this latter quantity is a dimensionless version of the spatial coordinate x which incorporates some nonlinear scaling via the relationship

$$\xi = \int_0^x \frac{\bar{m}\bar{C}_p}{\bar{\lambda}} \mathrm{d}x, \tag{4}$$

where $\bar{m}$, $\bar{\lambda}$ and $\bar{C}_p$ are respectively the constant dimensional mass flux per unit area, the variable dimensional thermal conductivity and the dimensional frozen specific heat at constant pressure; for simplification the latter has been assumed constant in the derivation of (1) and (2) and hence must be so considered in (4).

The foregoing model of the combined effects of flow compressibility and chemical energy release has recently been employed by Clarke (1983) in a discussion of the changes that take place in the structure of the field as M^2/θ^s increases from those low values, of order $\theta^2 \exp(-\theta/T_b)$, where T_b is the temperature of the burnt gas, that are typical of conventional thermal flames, up to $\theta^N \exp(-n\theta)$, where $0 < n < 1/T_b$. For Mach numbers in this latter category the field structure is independent of the effects of diffusion to a first order of accuracy. Each isolated fluid element experiences the effects of combustion in the form of a Semenov type of explosion process as it is convected through the region $\xi > 0$. This lack of dependence upon the behaviour of neighbouring fluid elements is in part due to the absence of diffusion as a process of first-rank importance, but also in part due to the absence of the effects of compressibility. In the analysis just referred to this latter is a consequence of the restriction to asymptotically small Mach numbers in the limit as $\theta \to \infty$.

The present work relaxes these restrictions on M and considers it throughout to be a quantity in the order class unity in the limit as $\theta \to \infty$. Thus compressibility

† The conventional 'thermal' flame is characterized by a Mach number $M \to 0$ that is exponentially small for large activation energy values ($\theta \gg 1$), so that T_s is the same as T for all practical purposes. The momentum equation (2) is then satisfied by its singular solution $T/u = 1$ and, as explained in Appendix B, this is equivalent to the usual isobaric assumption for thermal flames that normalizes p to unity throughout the flow field; (1) simplifies and becomes a single equation for T alone (e.g. Buckmaster & Ludford 1982).

effects now have a primary role to play. As pointed out in the article just referred to, the quantity within the { } brackets in (1) is the local Damköhler number, which, in view of the $O(1)$ character of M, will always be $o(1)$ in the $\theta \rightarrow \infty$ limit. As a consequence the sole combination of the effects of diffusion and reaction can never be strong enough to govern the field, even locally, and 'flame sheets' of the type that are found in low-speed thermal flames do not exist in these high-Mach-number flows.

Related work by Kapila, Matkowsky & Van Harten (1982) deals with a doubly infinite field, and, as a consequence, has to introduce the concept of an ignition temperature to circumvent the 'cold boundary difficulty'; these authors also employ the small-energy-release approximation. The present work is carried through without the need to adopt either of these features, and also deals with a greater range of gas-dynamical situations. The early study of the classical ZND type of Chapman–Jouguet detonation by Bush & Fendell (1971), using asymptotic methods, is highly relevant to the present, more general, class of problems. Their model required some modifications of strict Arrhenius kinetics but, where they are comparable, it will be shown in a forthcoming paper that the present analysis is in agreement with theirs.

It should be remarked that (2) has been derived from the momentum equation by eliminating the pressure in favour of the absolute temperature T and the gas velocity u (see appendix B, for example) and then integrating the equation once. It is the latter move that explicitly introduces the initial strain rate σ_0 into (2). Although it is a rather unusual parameter to find in problems of this general character it is nonetheless extremely convenient to carry out the whole of the present analysis on the assumption that σ_0 is a parameter open to selection, just like M, θ, etc. That σ_0 cannot be selected with total freedom will soon become evident, as will its relationship to quantities with more direct physical appeal such as the pressure difference across the half-space $(0 \leqslant \xi < \infty)$.

A few brief words about the organization of the paper may be helpful. Section 2 derives a very important result about the stagnation temperature, on which much subsequent work hinges. Section 3 discusses in general terms the nature of an equation for $u(\xi)$; the existence and broad structural character of the field is deduced from this relation. A number of subsidiary but necessary results are consigned to Appendices. Sections 4–7 describe general behaviour for all feasible inlet conditions. Finally, §§8–11 use large-activation-energy asymptotics to further expose and quantify the details of the flow-field behaviour.

2. Some general properties of solutions for T_s

Observing that $R > 0$ for all finite ξ, (1) shows that when T_s' is zero

$$T_s'' = -R \lesseqgtr 0 \quad (T_s \lesseqgtr T_{sb}).$$

Thus T_s must increase monotonically from its value of T_{s0},

$$T_{s0} = 1 + \tfrac{1}{2}(\gamma - 1)\, M^2, \tag{5}$$

up to the final value T_{sb} as ξ increases.

A formal solution of (1) makes

$$T_s' = -e^{\xi} \int_0^{\xi} e^{-\hat{\xi}}\, R\, d\hat{\xi} + e^{\xi} T_{s0}', \tag{6}$$

where

$$T'_{s0} = T'_0 + (\gamma-1) M^2 u'_0$$
$$\equiv q_0 + (\gamma-1) M^2 \sigma_0 \tag{7}$$

is the gradient of the stagnation temperature at $\xi = 0$, and (7) defines both q_0 and σ_0. Writing R_m for the mean value of R appropriate to the integral in (6), the latter can be re-expressed as follows:

$$T'_s = R_m + (T'_{s0} - R_m)\, e^{\xi}.$$

But T_s and T'_s must not be allowed to grow without bound as $\xi \to \infty$, and so

$$T'_{s0} = R_{m\infty} = \text{e.s.t.} > 0, \tag{8}$$

where $R_{m\infty}$ is the mean value of R_m as ξ increases. In view of the character of R as disclosed in (1) it is clear that R is always an exponentially small term (or e.s.t.) and positive. When the mixture is chemically inert, so that $R \equiv 0$, the only physically admissible solution of (1) requires $T_s = T_{s0}$ and $T'_{s0} = 0$. It will be seen that the modifications due to chemical energy release require an exponentially small amount of heat to be conducted into the flameholder *in excess* of any requirements on q_0 that may exist by reason of the gasdynamics of the chosen system (essentially, the value of σ_0; cf. (7) and (8) and note the exponentially small values of q_0 that occur for convected explosion flames; Clarke 1983).

Since T'_{s0} must be small under all circumstances, given that T_s must be bounded on physical grounds, it follows that (1) can always be written in the approximate form $T'_s \simeq R$, and $T''_s \ll T'_s$. Then the asymptotic estimate of T'_{s0} is that it is essentially equal to R when $T = 1$ and $T_s = T_{s0}$.

3. The equation for u

By eliminating T between (2) and (3) we can derive the following equation for u as a function of ξ:

$$\Gamma u u' = u^2 - (f - \hat{\sigma})\, u + f - 1 + \hat{F}(\xi). \tag{9}$$

It is convenient here and in the subsequent work to define the following quantities:

$$\hat{\sigma} \equiv \frac{2\gamma\sigma_0}{\gamma+1} \equiv \Gamma\sigma_0, \tag{10), (11}$$

$$f \equiv \Gamma\left\{1 + \frac{1}{\gamma M^2}\right\}, \tag{12}$$

$$F \equiv \tilde{Q}[f - \Gamma], \quad \hat{F}(\xi) \equiv \frac{\Gamma[T_s(\xi) - T_{s0}]}{\gamma M^2}, \tag{13), (14}$$

$$\tilde{Q} \equiv \hat{Q} - T'_{s0}, \tag{15}$$

where $\hat{Q}$ is the chemical energy released per unit mass of mixture, measured in units of $\bar{C}_p \bar{T}_0$ ($\bar{T}_0$ is the dimensional inlet temperature). It has been shown in §2 that T'_{s0} is an e.s.t. whose value can be readily estimated: $\hat{Q}$ is essentially a quantity of order unity, by sensible physical hypothesis, so that $\tilde{Q}$ can be treated from now on as known.

A global energy balance (or a reinterpretation in present terminology of results, especially (34), in §4 of Clarke 1983) shows that the stagnation temperature T_{sb} in the final burnt state is given by

$$T_{sb} = T_{s0} + \tilde{Q}. \tag{16}$$

Then (5) and (12) show that

$$\frac{\Gamma T_{s0}}{\gamma M^2} = f - 1 > 0, \tag{17}$$

while (12), (13), (14) and (16) show that

$$\hat{F}(\xi \to \infty) \to F, \tag{18}$$

since $T_s(\xi \to \infty) \to T_{sb}$.

It is convenient to rewrite (9) in the following form:

$$\Gamma u u' = \{u - u_+(\xi)\}\{u - u_-(\xi)\}, \tag{19}$$

where $u_{\pm}(\xi)$ are the real roots of the quadratic expression on the right-hand side of (9). The values of $u_{\pm}(\xi)$ when $\xi = 0$ and $\xi \to \infty$ are especially significant, and are identified as follows:

$$u_{\pm}(\xi \to \infty) \equiv u_{b\pm}, \tag{20}$$

$$u_{\pm}(0) \equiv u_{i\pm}. \tag{21}$$

Since $\hat{F}(0)$ is zero the subscript i in (21) can refer to either 'initial' or 'inert' states; the latter statement arises from the fact that when $\hat{Q}$ is zero T_s is equal to T_{s0} everywhere, and T'_{s0} is also zero; thus $\tilde{Q}$ vanishes.

It is important to note that (9) and (19) etc. imply that

$$u_+ + u_- = f - \hat{\sigma}, \quad u_+ u_- = f - 1 + \hat{F}(\xi) = \frac{\Gamma T_s(\xi)}{\gamma M^2}. \tag{22}$$

Thus

$$u'_+ = -u'_-, \tag{23}$$

$$u'_+ u_- + u_+ u'_- = \frac{\Gamma T'_s(\xi)}{\gamma M^2} \geqslant 0, \tag{24}$$

where the inequality here follows from §2. There is no loss of generality in taking

$$u_+ \geqslant u_-, \tag{25}$$

whence it readily follows that

$$-u'_- = u'_+ \leqslant 0. \tag{26}$$

Furthermore, in view of this monotone character of $u_{\pm}$, it follows that

$$u_{i+} \geqslant u_+(\xi) \geqslant u_{b+}, \quad u_{i-} \leqslant u_-(\xi) \leqslant u_{b-}. \tag{27a, b}$$

This general qualitative behaviour of $u_{\pm}(\xi)$ is illustrated in figures 2 and 3.

Before discussing the character and import of the integral curves $u = u(\xi)$ that are implied by (19), some further observations must be made. The final equilibrium or burnt state is characterized by the vanishing of all spatial gradients, in particular u', as $\xi \to \infty$. Therefore, from the relation (18) for $\hat{F}$, the quantities $u_{b\pm}$ must be the roots (note (9) and (18)) of the quadratic

$$u^2 - (f - \hat{\sigma})\, u + f - 1 + F = 0. \tag{28}$$

Limitations on the admissible values of $\hat{\sigma}$ that are implicit in relation (28) are discussed in Appendix A, which shows that, in general, $\hat{\sigma}$ must be less than a value $\hat{\sigma}_-$ defined in (A 3). The value $\hat{\sigma}_-$ depends upon γ, the energy release $\tilde{Q}$ and, particularly, the inlet Mach number M. There is a special behaviour of $\hat{\sigma}_-$ for inlet Mach numbers between the upper and lower Chapman–Jouguet values: this is explained and quantified in Appendix A. The curves of $u_{b\pm}$ and $u_{i\pm}$ versus $\hat{\sigma}$ are illustrated in figure 1, together with certain other salient pieces of information.

Appendix C defines local Mach number and, particularly, the local sonic speed u_*.

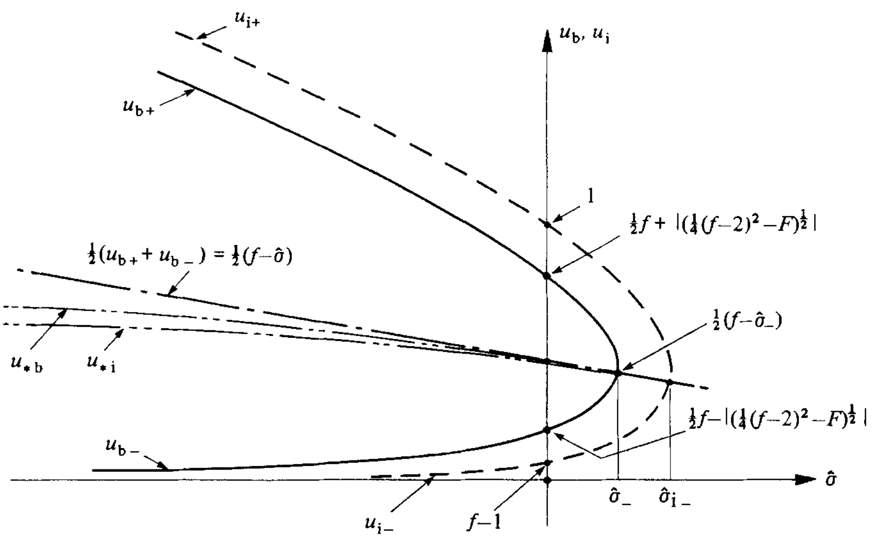

FIGURE 1. The loci u_{b+} and u_{b-}, of possible downstream conditions for a given $M > 1$ (or $f > 2$). The full line —— is the u_b locus for $F > 0$; the line --- for $F = 0$ describes the chemically inert case; when $M > 1$, u_{i+} is always equal to 1 when $\hat{\sigma} = 0$, since $f < 2$ for $M > 1$ (see 12)). The picture is identical for $M < 1$, except for the following important difference: since $f > 2$ for $M < 1$ (see 12)) the points 1 and $f-1$ are interchanged on the u_b axis, so that the initial condition $u = 1$ at $\xi = 0$ is now located in the neighbourhood of the u_{i-} locus. The quantity u^* is the sonic speed (see C7)), so that u_{*b} is its value in the burnt state at $\xi = \infty$, and u_{*i} is its value in the 'inert' or initial state at $\xi = 0$.

Evidently $u_+ > u_*$, while $u_- < u_*$; furthermore, u_* is a single-valued function of ξ and it is then useful to talk about domains of subsonic and supersonic flow. Sonic-speed loci appear in figures 1, 2 and 3. When $\hat{\sigma}$ has very small values, especially when u is near to the $u_+(\xi)$ locus, it is very helpful to know whether an integral curve is converging on u_+ or moving away from it. The necessary elementary analysis is described in Appendix D, which defines two loci, namely $u_{0\pm}$. These are sketched qualitatively on figure 3. Briefly, when an integral curve lies within the strips between u_{0+} and u_+ and u_{0-} and u_-, that integral curve is converging upon u_+; in the much larger space between u_{0+} and u_{0-} the integral curve is diverging from u_+.

There is now sufficient information to enable one to make a thorough analysis of the possible solution curves (i)–(v) that appear on figures 2 and 3. Note the behaviour of the pressure, described in Appendix B (particularly (B 3)), and the implications that p will change from unity at $\xi = 0$ to $p_{b\pm}$ as $\xi \to \infty$, where the latter pressures will depend weakly on $\hat{\sigma}$.

4. Solutions for supersonic inlet Mach numbers

Figure 1 indicates that $\hat{\sigma}$ may be chosen in the range $0 < \hat{\sigma} \leqslant \hat{\sigma}_-$. However, if $\hat{\sigma}$ is positive (19) requires

$$\Gamma\hat{\sigma} = \{1 - u_{i+}\}\{1 - u_{i-}\} > 0,$$

which implies that $1 > u_{i+}$ if $u = 1$ is to lie in the supersonic domain. Figure 2 then makes it clear that any integral curve that starts above the u_+ locus will remain above that locus and will continue with a positive slope; it will therefore never approach

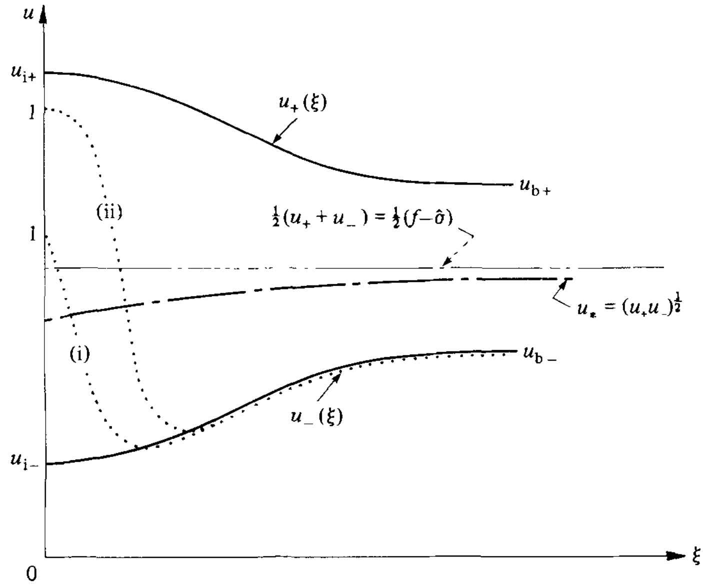

FIGURE 2. A sketch of the (u, ξ)-plane (*not* to scale). The variations in $u_\pm$ take place over exponentially long intervals in ξ, while the reduction in u along curve (i), for example, to its intersection with u_- only occupies an order unity interval in ξ, since the position of point $u = 1$, $\xi = 0$ relative to $u_{i\pm}$ is such as to imply that $\Gamma u'(0) = \hat{\sigma}$ is of order unity. For any given M and γ (or f) the solution curve (i) *and* $u_\pm(\xi)$ both depend upon $\hat{\sigma}$ and the picture will be different in detail for each $\hat{\sigma}$-value. Because there is no radical *qualitative* change in the form of the $u_\pm$ curves with changes in $\hat{\sigma}$, a second solution curve (ii) is illustrated on the present figure, simply to save space. Curve (ii) is for a smaller $\hat{\sigma}$-value, so that the hydrodynamic adjustment zone in the neighbourhood of $\xi = 0$ is more like a 'complete' shock wave; remember that in practice (i) is associated with one set of $u_\pm$ curves, reflecting the choice of $\hat{\sigma}$, while (ii) is associated with a different pair of $u_\pm$ loci.

a downstream equilibrium state ($u_{b\pm}$). It is not possible to construct an initially supersonic solution with $\hat{\sigma} = 0$, and thus $1 = u_{i+}$, for the same reason. Thus possible $\hat{\sigma}$ values must be negative (more will be said on this topic below) and integral curves (i) and (ii) on figure 2 illustrate the consequences of a pair of values of $\hat{\sigma}$ that could be labelled loosely as large and not-so-large, respectively. Such loose labels will be given more substance and quantification in the sections to follow on the asymptotic analysis.

When $u_+ > 1 \geqslant u > u_-$, (19) makes it clear that $u' < 0$. Thus both curves (i) and (ii) descend rapidly towards u_-, crossing the u_* sonic locus as they do so. It is clear that u' vanishes as u crosses u_- (note from Appendix D that, once below u_{0-}, the integral curve will already be converging on u_+); once below u_- the integral curves follow the u_- locus very closely. Curves (i) and (ii) therefore represent a rapid transition from supersonic to subsonic flow adjacent to the face of the holder at $\xi = 0$, followed by regions of much slower transition to the final equilibrium state, namely u_{b-} in this case.

The rapid-adjustment domain will subsequently be identified as an effectively chemically inert event of shock-wave (or part-shock-wave)-like character. The subsequent region will be identified as the one in which all of the chemical energy release (burning) takes place, within which 'diffusion' is relatively unimportant.

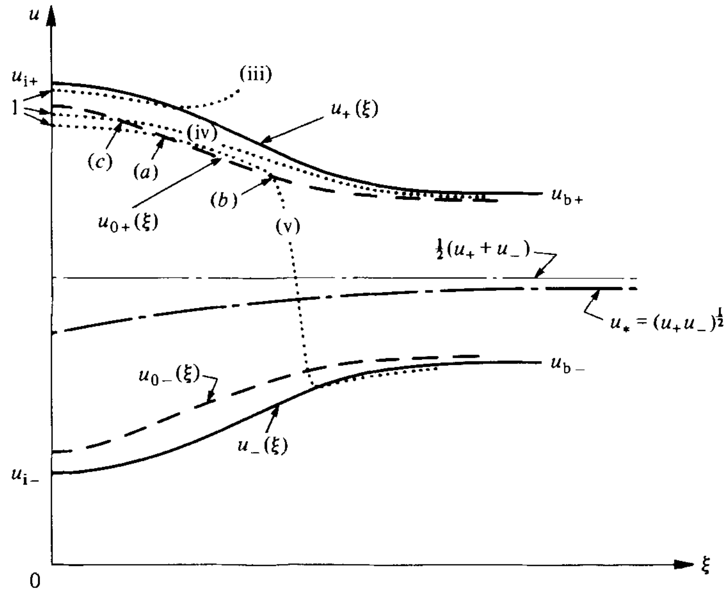

FIGURE 3. A sketch of the (u, ξ)-plane (*not* to scale) that illustrates the relative dispositions of $u_{\pm}$ and the lines $u_{0\pm}$ on which $U' = u' - u'_+ = 0$. Three possible integral curves are sketched for three different $\hat{\sigma}$-values on the same diagram in order to economize on space; strictly there should be different $u_{\pm}$ curves for each $\hat{\sigma}$ chosen, but they must all be qualitatively similar to the ones displayed in the figure. Curve (iii) starts between u_{i+} and $u_{0+}(0)$, and must therefore depart from u_+ in the manner shown; curve (iv) is captured by the u_{0+} locus at (c) and thereafter converges on u_+ to meet the latter at u_{b+} and thereby constitute a fully supersonic field. Curve (v) is also captured by u_{0+} at (a), but escapes from it at (b) and crosses the sonic line u_* via a local shock transition; the final stages towards u_{b-} are via subsonic flow.

5. Solutions for supersonic inlet Mach numbers; small $\hat{\sigma}$-values

When the chosen value of $\hat{\sigma}$ is very small, in a way that will be quantified in a later section, some special considerations arise in the case $M > 1$. The latter implies that $u = 1$ will be close to, but below, u_{i+}.

Referring to figure 3, consider the three curves (iii), (iv) and (v). Curve (iii) starts between u_{0+} and u_+; it therefore converges on u_+ and soon crosses it (with $u' = 0$) and escapes into the domain $u > u_+$. Curve (v) starts below u_{0+} and therefore diverges from u_+ until, at point (a), it is overtaken by u_{0+}. Between (a) and (b), curve (v) converges on u_+, but then, in view of the convergence of u_{0+} itself onto u_+, (v) emerges at point (b) and descends rapidly towards u_-, where it is captured and turned to a final subsonic transition to the solution at u_{b-}. Curve (iv) illustrates a transition from a point near to u_{0+}, and just below it at $\xi = 0$, to end finally at the supersonic solution point u_{b+}. Since the locus u_{0+} does not intersect u_*, and (iv) lies mostly between u_{0+} and u_+, the transition from first to final states is a fully supersonic one. Of course a diagram such as figure 3 strictly has $u_{\pm}$ drawn for a given $\hat{\sigma}$-value, which then determines both $T_s(\xi)$ and $u(\xi)$. Since the point at which an integral curve begins on the $\xi = 0$ line also determines $\hat{\sigma}$, only one of the illustrated curves can actually be the solution curve $u(\xi)$ or, at worst, none of them may have this property. From the continuous dependence of solutions $T_s(\xi)$ and hence $u_{\pm}(\xi)$ and $u(\xi)$ on the parameter $\hat{\sigma}$, at least in $\hat{\sigma} < 0$, it follows that one can always find one $\hat{\sigma}$-value for which the solution looks like (iv) and another different $\hat{\sigma}$-value for which the solution looks like (v). From the general qualitative character that the integral and other curves have it can be seen that these $\hat{\sigma}$ (or σ_0) values will be very small (of order u'_+ itself) and negative.

Support for the foregoing descriptions of solution curve behaviour will be found in the asymptotic analysis to be described below, particularly in §10.

To summarize, the conclusions about general features of the field for a supersonic inlet flow are as follows. When the $\hat{\sigma}$ (or σ_0) value is large, that is to say *not* in the neighbourhood of $u'_+(0)$, the field begins with a rapid zone of near-shock-like character within which the local Mach number is brought to a subsonic value with effectively no intervention from combustion-energy release; the latter process is completed in a wholly subsonic condition but with steady increase of local Mach number. Examples are provided in curves (i) and (ii) on figure 2. When $\hat{\sigma}$ diminishes towards values in the neighbourhood of $u'_+(0)$, the shock moves away from $\xi = 0$, and, as illustrated by curve (v) on figure 3, can occur *after* some substantial amount of combustion activity has taken place in the upstream supersonic parts of the flow.

For the fully supersonic field, as well as for those fields that contain a shock at intermediate locations far from $\xi = 0$, $\hat{\sigma}$-values are extremely small, so that final pressures are very close to the values obtained from (B 4), with $u_{b\pm}$ given by (A 1) with $\hat{\sigma} = 0$, namely

$$p_{b\pm}|_{id} = \tfrac{1}{2}M^2\{f \mp \gamma|[(f-2)^2-4F]^{\frac{1}{2}}|\}. \tag{29}$$

The subscript id stands for an 'ideal', $\hat{\sigma} = 0$, value.

An important factor in the establishment of a supersonic combustion field or, equally, a field with a shock far downstream of the inlet plane $\xi = 0$, will be its ultrasensitivity to the maintenance of a pressure difference across the domain very close to 1 to $p_{b\pm}|_{id}$ (defined in (29)) and to within extremely narrow limits.

6. Solutions for subsonic inlet Mach numbers

The first significant difference between the present case, $M < 1$, and the $M > 1$ situation described in §§4, 5 is encountered in the details of the admissible downstream conditions, as can be seen from figure 1. Shifting the initial point $u = 1$, $\xi = 0$, onto the u_{i-} locus (see caption to figure 1) has the immediate effect of permitting $\hat{\sigma}$ to have positive values, so long as they are less than $\hat{\sigma}_-$. Equation (19) still describes the behaviour of $u(\xi)$, of course; in the present circumstance a condition $1 < u_{i-}$ gives rise to positive values of u'; consultation of either of figures 2 or 3 shows that continuation of this state of affairs is wholly consistent with the final state $u(\xi \to \infty) \to u_{b-}$ and a continuous field structure exists.

Since $1 < u_{*i} \leqslant u_{*b} \leqslant u_+$ it can be seen from figures 2 or 3 that *any* value of $\hat{\sigma} \leqslant 0$ will also lead to a solution curve that remains in the neighbourhood of u_-. In particular, if $|\hat{\sigma}|$ is of order unity, which implies that $|(u_{i+}-1)(1-u_{i-})| = O(1)$, the solution curve moves quickly from $u = 1$ at $\xi = 0$ into a close proximity to the u_- locus; the hydrodynamic adjustment zone is therefore *always* adjacent to $\xi = 0$ when $M < 1$.

Since any integral curve that finds itself below the u_* locus has no way of increasing to pass through u_* and therefore converge on u_+, a transition from subsonic to supersonic flow is absolutely forbidden. This result should be compared with the similar impossibility of finding a structure for the class of strong deflagrations in Hugoniot-style analysis of metastable streams emanating from $\xi = -\infty$ (e.g. Williams 1965, §6.2; for a discussion from a somewhat different standpoint see Clarke 1980, §4.5).

To summarize, there is a continuous field structure for all subsonic inlet Mach numbers for all $\hat{\sigma} \leqslant \hat{\sigma}_-$ (see (A 3)). Any largely hydrodynamic adjustments, in which combustion plays a minor role, are always adjacent to $\xi = 0$.

7. Inlet Mach number between the CJ values

When $M_{CJ-} < M < M_{CJ+}$, (A 7) shows that $\hat{\sigma} \leqslant \hat{\sigma}_- < 0$. If $\hat{\sigma}_-$ is not too close to zero, and it will be shown in § 10 to follow what the precise meaning of 'too close' must be, the field must *begin* with a zone of rapid gasdynamical adjustment under nearly chemically inert conditions. Thus u will diminish in such a zone towards the 'inert' value u_{i-}, where

$$u_{i-} = \tfrac{1}{2}(f-\hat{\sigma}) - |\{\tfrac{1}{4}(f-\hat{\sigma})^2-(f-1)\}^{\frac{1}{2}}|, \tag{30}$$

as can be seen from (A 1) with $F = 0$.

Suppose for the present that $\hat{\sigma}$ is exactly equal to $\hat{\sigma}_-$; then (A 3) shows that

$$\tfrac{1}{2}(f-\hat{\sigma}_-) = (f-1+F)^{\frac{1}{2}}, \tag{31}$$

and combination of (30) and (31) makes

$$u_{i-} = (f-1+F)^{\frac{1}{2}} - F^{\frac{1}{2}}. \tag{32}$$

From the general result that

$$T_s = M^2u^2\{m^{-2}+\tfrac{1}{2}(\gamma-1)\} = T\{1+\tfrac{1}{2}(\gamma-1)\,m^2\},$$

where m is the local Mach number, defined in (C 1), one can now find m_{i-} and T_{i-} by using the condition $T_s = T_{si}$ and taking the value of u_{i-} from (32). After a certain amount of tedious but straightforward algebra it can be shown that

$$(1-m_{i-}^2)^2 = 2(\gamma+1)\,\frac{\tilde{Q}m_{i-}^2}{T_i}. \tag{33}$$

Comparing (33) with (A 5), with $\leqslant$ replaced by $=$, it can be seen that m_{i-} under the condition $\hat{\sigma} = \hat{\sigma}_-$ is a lower CJ Mach number for flow in the subscript-(i$-$) state. Note that the heat-addition term must be made non-dimensional with respect to the local temperature; hence the appearance of T_{i-} in (33).

For any $\hat{\sigma} < \hat{\sigma}_-$ the associated m_{i-} will be less than the lower CJ value given in (33), and it can be seen that a steady-state solution for inlet Mach numbers in the interval between M_{CJ-} and M_{CJ+} is only possible with the immediate appearance of a thin gasdynamical adjustment zone that reduces the local Mach number to a *local* lower CJ value, or less.

Some further information about these states of affairs can be acquired from an asymptotic analysis of the field in the limit as $\theta \to \infty$.

8. Asymptotic analysis: nearly inert regions near $\xi = 0$

The foregoing sections have established a number of important results of a qualitative character, and it is now time to expand a little on the details of this character and to give it, at least approximately, some quantitative form. The method of large-activation-energy asymptotics is particularly well suited to these ends and it is this method that will be employed from here on.

When $\hat{\sigma}$ (or $\hat{\sigma}_0$) is large it has been shown in several places that the field must begin with a zone of almost chemically inert gasdynamical variations in the neighbourhood of $\xi = 0$. Since $u_{\pm}(\xi)$ are equal to $u_{i\pm}$ to a first order of accuracy in this neighbourhood, (19) shows that a first estimate of u, where

$$u = u(\xi;\theta) \sim u_i(\xi), \tag{34}$$

is given by

$$\Gamma u_i u_i' = (u_i - u_{i+})(u_i - u_{i-}), \tag{35}$$

with the condition $u_i(0) = 1$. In the absence of reaction effects, (35) is expressive of a balance between convective and diffusive effects only; in the terminology of the paper on low-speed combustion (Clarke 1983) it represents a CD region. The solution is most easily given in implicit form as

$$u_{i+}\ln\left\{\frac{|u_i - u_{i+}|}{|1-u_{i+}|}\right\} - u_{i-}\ln\left\{\frac{|u_i - u_{i-}|}{|1-u_{i-}|}\right\} = (u_{i+} - u_{i-})\frac{\xi}{\Gamma}. \tag{36}$$

Since $u_{i+} - u_{i-} > 0$, it can be seen that $u_i \to u_{i-}$ as $\xi \to +\infty$, so that the initial inert, or nearly inert, transition is essentially one to subsonic regions near the u_- locus. When a result such as (36) is valid, T_s is essentially equal to T_{s0} to a first order, so that T' and u' will behave in closely related ways. One must therefore begin to suspect that, as ξ increases and u', T' both diminish as $u_i \to u_{i-}$, the R-term in (1) can no longer be neglected, as it has been, implicitly, in the development of the result (36). For obvious reasons such a domain has been called a CDR region. To test this hypothesis write

$$T \sim T_{i-} + g_i \tau_i(\xi - \xi_i), \quad u \sim u_{i-} + g_i U_i(\xi - \xi_i), \tag{37}$$

where g_i is a gauge function of θ that is to be determined, and ξ_i is a value of ξ within the new domain. Without loss of generality it can be assumed that

$$U_i(0) = 1. \tag{38}$$

The distance ξ_i is to be determined by matching the proposed solution with (36).

Substituting (37) into (9) and (14) shows that

$$\Gamma u_{i-} U_i' - [2u_{i-} - (f - \hat{\sigma})]\, U_i = \frac{2}{(\gamma+1) M^2}\tau_{si}, \tag{39}$$

where

$$\tau_{si} = \tau_i + (\gamma - 1) M^2 u_{i-} U_i. \tag{40}$$

The $O(1)$ terms that appear during the course of this substitution of (37) into (9) and (14) vanish because T_{i-} and u_{i-} are defined so that

$$T_{i-} + \tfrac{1}{2}(\gamma - 1) M^2 u_{i-}^2 = T_{s0}. \tag{41}$$

Substituting (37) into (1), and using the definition (40), shows that

$$\tau_{si}' - \tau_{si}'' = 1 \tag{42}$$

provided that

$$g_i = A_4 \theta^s M^{-2} e^{-\theta/T_{i-}} \{T_{sb} - T_{s0}\}. \tag{43}$$

Furthermore this estimate of the influence of R remains valid only so long as

$$\theta g_i T_{i-}^{-2} \tau_i = o(1). \tag{44}$$

The general solution of (42), with two so-far-arbitrary constants C and D, is given by

$$\tau_{si} = 1 + (\xi - \xi_i) + C + D\exp(\xi - \xi_i), \tag{45}$$

and we encounter a matter that presents itself in other places in the asymptotic analysis, namely the existence of a term in the solution for T_s (and hence, here, τ_{si})

that grows without bound as ξ increases. Such behaviour is not acceptable in the whole domain of ξ, nor is it any more acceptable in the various asymptotic domains into which the θ-limit divides the whole domain. It is therefore necessary to make $D = 0$, with the result that

$$\tau_{si} = \hat{C} + (\xi - \xi_i), \quad \hat{C} \equiv 1 + C, \tag{46}$$

and (39) can now be solved to find U_i.

The solution of (39) is expedited by remembering that

$$f - \hat{\sigma} = u_+ + u_- = u_{i+} + u_{i-}$$

(see (22)), so that

$$-[2u_{i-} - (f - \hat{\sigma})] = (u_{i+} - u_{i-}) > 0.$$

Defining

$$A \equiv (u_{i+} - u_{i-})/\Gamma u_{i-} > 0, \tag{47}$$

$$a \equiv \frac{2}{(\gamma+1)\Gamma M^2 u_{i-}} = \frac{1}{\gamma M^2 u_{i-}}, \tag{48}$$

the solution for U_i that obeys (38) is

$$U_i = \frac{a}{A}(\hat{C} + (\xi - \xi_i)) - \frac{a}{A^2} + \frac{a}{A}\left(\frac{1}{A} - \hat{C}\right) e^{-A(\xi - \xi_i)} + e^{-A(\xi - \xi_i)}. \tag{49}$$

In order to match (37) with the solution (34) and (36), ξ_i is chosen so that

$$A\xi_i = \frac{\theta}{T_{i-}} - \ln(A_4 M^{-2}\theta^s(T_{sb} - T_{so})), \tag{50}$$

and $\hat{C}$ is chosen so that

$$|1 - u_{i-}| \left\{\frac{u_{i+} - u_{i-}}{|1 - u_{i+}|}\right\}^{u_{i+}/u_{i-}} = 1 + \frac{a}{A^2} - \frac{a\hat{C}}{A}. \tag{51}$$

A solution such as (46) is founded on the hypothesis that constants such as $\hat{C}$ are $O(1)$ when the relevant coordinate $\xi - \xi_i$ is also $O(1)$; it is therefore essential to incorporate $\ln(\theta^s)$ in the value for ξ_i; the associated order-unity numbers $A_4 M^{-2}(T_{sb} - T_{s0})$ are not essential in (50), but it is somewhat tidier to have them there, and it simplifies the solution for $\hat{C}$ to first order.

The foregoing results make the tacit assumption that $|1 - u_{i-}|$ is $O(1)$. But §6 explains that subsonic solutions exist for all $\hat{\sigma} \leqslant \hat{\sigma}_-$, so that there is certainly an admissible range of very small $\hat{\sigma}$-values. The relationship (35) shows that

$$|\hat{\sigma}| = |1 - u_{i+}|\,|1 - u_{i-}|,$$

and when $|\hat{\sigma}|$ is small it follows that $|1 - u_{i-}|$ is also very small.

Suppose that

$$|1 - u_{i-}| = e^{-n\theta}, \quad n > 0; \tag{52}$$

then ξ_i as given in (73) should be modified to read

$$A\xi_i = \frac{\theta}{T_{i-}} - n\theta - \ln(A_4 M^{-2}\theta^s(T_{sb} - T_{so})). \tag{53}$$

When $u_{i-} = 1 \pm e^{-n\theta}$, as is implied by (53), it is clear from (41) that T_{i-} is likewise of order $1 \mp O(e^{-n\theta})$. Thus (53) contains a term $(1-n)\,\theta$, and ξ_i will only lie in the centre of a 'corrective' CDR domain, in which R begins to become prominent, that is well downstream of $\xi = 0$ if $n < 1$. To be a little more precise it can be seen that the inert CD adjustment zone is only required if

$$\hat{\sigma} > O(A_4 M^{-2}\theta^s e^{-\theta}(T_{sb} - T_{s0})). \tag{54}$$

This criterion defines the word large as used at the outset of this section. When $\hat{\sigma}$ is of the same order as the right-hand side of (54) a modification of the present analysis is required and will be undertaken in §9.

In the meantime it is important to assess behaviour at the downstream ($\xi - \xi_i \to \infty$) end of the CDR transition layer whose character is described by (37), (43), (46), (49), (50) and (51). In the first place one can see that T_s and u behave like

$$T_s \sim T_{si-} + g_i[\xi - \xi_i],$$

$$u \sim u_{i-} + g_i \frac{a}{A}[\xi - \xi_i],$$

to first order when $\xi - \xi_i$ is large, from which it follows that

$$T \sim T_{i-} + g_i \frac{a}{A^2}(\gamma - 1) M^2 u_{i-} + g_i \left\{1 - (\gamma - 1) M^2 u_{i-} \frac{a}{A}\right\} [\xi - \xi_i]$$

under the same conditions.

The various constant factors a, A, etc. in these expressions can be re-organized as follows. From (22) one can find the relation

$$u_{i+} = \frac{2T_{i-}}{(\gamma + 1) M^2 u_{i-}} + \frac{(\gamma - 1) u_{i-}}{\gamma + 1},$$

so that, noting (47),

$$u_{i+} - u_{i-} = \frac{2u_{i-}}{\gamma + 1}\left(\frac{1}{m_{i-}^2} - 1\right) = \Gamma u_{i-} A,$$

where

$$m_{i-} = M u_{i-} (T_{i-})^{-\frac{1}{2}}$$

is the local Mach number under subscript-(i −) conditions. After some manipulations it can finally be shown that

$$\frac{u}{u_{i-}} \sim 1 + \frac{g_i}{T_{i-}} \frac{m_{i-}^{-2}}{m_{i-}^{-2} - 1} [\xi - \xi_i] - \frac{g_i}{T_{i-}} \frac{\gamma m_{i-}^{-2}}{(m_{i-}^{-2} - 1)^2}, \tag{55}$$

$$\frac{T}{T_{i-}} \sim 1 + \frac{g_i}{T_{i-}} \frac{1 - \gamma m_{i-}^2}{1 - m_{i-}^2} [\xi - \xi_i] + \frac{g_i}{T_{i-}} \frac{\gamma(\gamma - 1)}{(m_{i-}^{-2} - 1)^2} \tag{56}$$

as $\xi - \xi_i$ becomes large and positive. These results will prove to be interesting in the light of developments in §9.

Finally in this section it must be observed that the requirement for bounded variation of T_s, such as leads to solutions like (46) for example, is consistent with the notion, expressed at the end of §2, that T_s' is equal to R to first order. The implication is that

$$T_{s0}' \approx A_4 \theta^s M^{-2} e^{-\theta} (T_{sb} - T_{s0}), \tag{57}$$

and it can be seen from (15) and (16) that it is sufficiently accurate to make T_{sb} in (57) equal to $T_{s0} + \hat{Q}$.

9. Asymptotic analysis near $\xi = 0$ for small $\hat{\sigma}$

What constitutes a small value of $\hat{\sigma}$ (or σ_0) has been ascertained in §8, in particular in (54) *et seq.*, from which it can be seen that we should now consider the particular case for which

$$\sigma_0 = g_i \tilde{\sigma}_0, \quad g_i \equiv A_4 M^{-2} \theta^s e^{-\theta} (T_{sb} - T_{s0}), \tag{58}$$

where $\tilde{\sigma}_0$ is $O(1)$ by hypothesis.

As soon as one appreciates that the gauge factor g_i is precisely the same as R in (1) evaluated when $T = 1 = u$, it can be seen that it is now necessary to seek asymptotic solutions in the form

$$\left.\begin{aligned} u &\sim 1+g_i U_i(\xi), \\ T &\sim 1+g_i \tau_i(\xi) \end{aligned}\right\} \tag{59}$$

which describe a CDR domain in the neighbourhood of $\xi = 0$. The boundary conditions demand that

$$U_i(0) = 0 = \tau_i(0). \tag{60}$$

Equations (58) and (59) in (1)–(3) give, in the limit as $\theta \to \infty$ with ξ fixed,

$$U_i' = U_i + \frac{1}{\gamma M^2}(\tau_i - U_i) + \tilde{\sigma}_0, \tag{61}$$

$$\tau_{si}' - \tau_{si}'' = 1, \quad \tau_{si} = \tau_i + (\gamma - 1) M^2 U_i. \tag{62), (63}$$

The solution of (62) that satisfies (60) is

$$\tau_{si} = \xi + (\tau_{si0}' - 1)(e^{\xi} - 1), \tag{64}$$

where τ_{si0}' is the derivative of τ_{si} with respect to ξ at $\xi = 0$.

Now τ_{si}' can only remain bounded if τ_{si0}' is equal to unity. In the circumstances

$$\tau_{si} = \xi, \tag{65}$$

and the resultant value for T_{s0}' is exactly the same as the value given in (57), as indeed we should expect it to be. Combination of (63) and (65) leads to the elimination of τ_i from (61), and the resulting equation for U_i can be solved, with the aid of (60), to give

$$U_i = -\frac{M^{-2}}{\gamma\phi}\xi + \left(\frac{M^{-2}}{\gamma\phi} + \tilde{\sigma}_0\right)\frac{1}{\phi}(e^{\phi\xi} - 1) \tag{66}$$

together with

$$\tau_i = \frac{1}{\phi}\left(1 - \frac{1}{\gamma M^2}\right)\xi - \frac{(\gamma-1)M^2}{\phi}\left(\frac{1}{\gamma M^2 \phi} + \tilde{\sigma}_0\right)(e^{\phi\xi} - 1). \tag{67}$$

The quantity ϕ is defined by

$$\phi = \frac{1}{\gamma}(1 - M^{-2}), \tag{68}$$

so that

$$\phi \gtrless 0 \Leftrightarrow M \gtrless 1. \tag{69}$$

First consider the subsonic case $M < 1$, $\phi < 0$. The exponential terms in (66) and (67) decay as ξ increases, and U_i therefore always ultimately increases linearly with ξ. The temperature also ultimately varies linearly with ξ, but only increases with ξ when $\gamma M^2 < 1$. When the inlet flow speed is greater than the isothermal sound speed at $\xi = 0$ (i.e. when $M > 1/\gamma^{\frac{1}{2}}$) the temperature falls as ξ increases. Thus the energy that is liberated by the chemical reaction goes into increasing the kinetic energy of the gas flow, which expands in the process, so that energy is even extracted from the thermal mode of energy storage, and the local temperature diminishes. All of these effects are present near $\xi = 0$ although they are of small amplitude by virtue of the gauge factor g_i in (59).

Before turning to supersonic inlet conditions, $M > 1$, it is instructive to compare (59) (with (66) and (67)) with results (55) and (56) in §8. Note first that if one replaces M^2 in ϕ in (68) by m_{i-}^2, the constant factors in (66) and (67) become exactly equal to those found on (55) and (56), *provided* that $\tilde{\sigma}_0$ is zero. Indeed, when $\tilde{\sigma}_0$ has this value, the behaviour of u/u_{i-} and T/T_{i-} as $\xi-\xi_i$ increases becomes exactly the same as the behaviour of u and T in the present case, since g_i/T_i is the appropriate renormalization of the gauge factor consistent with the fact that T and u are now measured in units of temperature and velocity at the subscript-(i−) condition. The significance of this is clear; when $|\sigma_0|$ is large, CD-type gasdynamical adjustments occur, ultimately with some explicit intervention from the combustion reaction in a CDR domain, and smooth out any σ_0 inputs so that the flow proceeds, somewhat downstream of ξ_i (see (53)), in a manner that is both subsonic and essentially independent of σ_0. Since ξ_i is necessarily large it follows that the influence of σ_0 is lost only at some substantial distance downstream of the holder. 'Substantial' here means a large number of diffusion lengths (see (4)) by which one means a length equal to

$$\left\langle \frac{\bar{\lambda}}{\bar{m}\bar{C}_p} \right\rangle = \left\langle \frac{\bar{\lambda}/\bar{\rho}\bar{C}_p\bar{a}}{m} \right\rangle. \tag{70}$$

The $\langle\ \rangle$ brackets define an average value, and $\bar{a}$ is the local frozen sound speed; in present (near $\xi = 0$) circumstances it is sufficient to evaluate quantities in (70) under subscript-(i−) conditions. The quotient $\bar{\lambda}/\bar{\rho}\bar{C}_p\bar{a}$ defines a molecular mean free path, so that in dimensional terms ξ_i is of magnitude

$$\frac{\theta}{AT_{i-}}\frac{l_{i-}}{m_{i-}} = \gamma\theta l_{i-}(m_{i-}^{-1}-m_{i-})^{-1}T_{i-}^{-1}, \quad m_{i-} < 1, \tag{71}$$

where l_{i-} is the free path and A has been eliminated by using the result towards the end of §8. All of the CD- and CDR-type gasdynamical adjustments therefore take place within a *dimensionally* thin layer adjacent to $\xi = 0$; quite naturally, such layers are of shock-wave-like widths.

The asymptotic analysis in §8 and the present section has so far dealt with either the subsonic situation, described in general terms in §6, the 'in-between' CJ case discussed in §7, or with supersonic solutions of types (i) and (ii), shown on figure 2, in §5. When σ_0 is small, as in (58), *and* $M > 1$ the situation changes and deserves some special treatment. This can be appreciated from the fact that $\phi > 0$ when $M > 1$ (see (69)) and that U_i and τ_i in (66) and (67) contain terms in $\exp(\phi\xi)$. The situation is evidently of the kind illustrated in general terms in curves (iii), (iv) and (v) of figure 3.

10. Asymptotic analysis for small σ and $M > 1$

Suppose for the moment that the flow field is chemically inert, so that (36) is the complete solution of the problem. When σ_0 is very small and negative, so that u_i is therefore very close to u_{i+}, this complete solution shows that

$$u_i \approx 1+\frac{\hat{\sigma}}{1-u_{i-}}\left\{\exp\frac{\Gamma(u_{i+}-u_{i-})\,\xi}{u_{i+}}-1\right\} \tag{72}$$

in the neighbourhood of $\xi = 0$. In particular, if σ_0 is given by (58), (72) can be rewritten in the form

$$u_i \approx 1+g_i\tilde{\sigma}_0\phi^{-1}(e^{\phi\xi}-1), \quad \gamma > 0, \tag{73}$$

where ϕ is defined in (68) and the translation from $u_{i\pm}$ terminology in (72) into terms of ϕ is made with the aid of relations (10), (12) and (A 1), plus the fact that $\hat{\sigma} = O(g_i) = o(1)$.

Comparing (73) with (66) shows that the part of U_i that is proportional to $\tilde{\sigma}_0$ is describing behaviour in the upstream outskirts of an inert-flow shock wave, provided $\tilde{\sigma}_0$ is negative. The remaining parts of U_i in (66) can have their origin traced back to the existence of combustion-energy release by the reaction term R in (1), the linear superposition of reactive and inert behaviour being validated by the small-perturbation character of the asymptotic solutions (59).

The comparisons just made are important because they show that the apparently unbounded exponential-growth terms in (66) and (67) are perfectly proper assessments of possible physical behaviour, *unlike* $\exp(\xi)$ terms in τ_{si} in (64) for example. The seeming unbounded behaviour of the $\exp(\phi\xi)$ terms will be checked by a re-assessment of conditions in the breakdown domain, where $g_i U_i$ ceases to be $O(g_i)$, and the establishment of a new CD type of asymptotic domain, in which (35) will clearly play a major role. At least, these remarks will be true, in light of (66) and, especially, of the general forms of solution (iii), (iv) and (v) exhibited on figure 3, provided that $\tilde{\sigma}_0$ obeys a more stringent condition in the reactive case than a mere requirement for it to be negative.

Remembering that $\phi > 0$ when $M > 1$, (66) shows that when

$$0 > \tilde{\sigma}_0 > \frac{-1}{\gamma M^2 \phi} = \frac{-1}{M^2 - 1}, \tag{74}$$

U_i, and hence u, ultimately *increases* with increasing ξ. When ξ is small, U_i behaves like

$$U_i \sim \tilde{\sigma}_0 \xi$$

to first order, and therefore u begins by actually *de*creasing near $\xi = 0$. Clearly this is the behaviour exhibited by (iii) on figure 3, and if $\tilde{\sigma}_0$ is in the range described by (74) it must be anticipated that, as for curves (iii), no solutions exist.

If

$$\tilde{\sigma}_0 < \frac{-1}{\gamma M^2 \phi} = \frac{-1}{M^2 - 1} \equiv \tilde{\sigma}_{0\,\mathrm{crit}} \tag{75}$$

the exponential, specifically $\exp(\phi\xi)$, reductions in u are a herald of imminent shock behaviour, such as that illustrated in (v) on figure 3. One must, however, exercise some caution here because (62) is only valid so long as

$$\theta g_i \tau_i = o(1). \tag{76}$$

This criterion mimics (44), but for the special circumstance that $T_{i-} \simeq 1 \simeq u_{i-}$. If

$$\tilde{\sigma}_0 + \frac{1}{\gamma M^2 \phi} < 0 \tag{77}$$

is $O(1)$ it is evident that (76) will fail first as a result of the behaviour of the $\exp(\phi\xi)$ term and not as a result of the term that is linear in ξ (see (67)). It will be seen shortly that failure of the estimates (59) as a result of the *linear-term* behaviour leads into a domain of growing chemical activity (the induction zone); failure of (59) from the $\exp(\phi\xi)$ origins leads to a nearly inert shock transition to the vicinity of the u_- locus, with the result that the induction zone is then initiated under subsonic flow conditions. To this extent the situation implicit in the circumstances described by (77) *et seq.* is really of the type illustrated in curve (ii) on figure 2.

When (77) is true, but the difference is now not $O(1)$ in the θ-limit but $o(1)$, it can be appreciated that the shock-like descent of the $u(\xi)$ solution curve towards u_- will be delayed in ξ, even to such an extent that the induction zone will begin to develop long before the shock appears; this is the situation implicit in curve (v) on figure 3.

But if $\tilde{\sigma}_0 + 1/\gamma M^2\phi$ is $o(1)$ as $\theta \to \infty$, it should not appear in U_i, since there will in general be additional terms in the asymptotic representations (59) that are of comparable or even larger magnitude.

The asymptotic analysis described here is therefore not precise enough to yield full information on the field behaviour for $\tilde{\sigma}_0$ values that approach $\tilde{\sigma}_{0\,\mathrm{crit}}$ (see 75)) arbitrarily closely from below. To put this statement another way, and indeed to thereby encourage a more constructive outcome of the foregoing analysis, there is now enough information contained above to validate the notion that a nearly inert shock can exist anywhere in the field, even in the parts where reaction is proceeding apace, but we are unable to associate a precise value of $\tilde{\sigma}_0$ with a given shock position (other than to remark that it must be such as to make $\tilde{\sigma}_{0\,\mathrm{crit}} - \tilde{\sigma}_0 > 0$ and $o(1)$) without proceeding to asymptotic estimates that are of higher accuracy than (effectively) one-term expansions, such as (59).

Therefore, if one is prepared to forgo more precise knowledge about the difference $\tilde{\sigma}_{0\,\mathrm{crit}} - \tilde{\sigma}_0$ than that it must be positive and $o(1)$ in the θ-limit, one can propose the following simple way to illustrate behaviour of type (v) on figure 3. It can be assumed that an effectively inert shock wave whose velocity profile obeys (35) to first order exists and is centred about $\xi = \xi_{\mathrm{sh}}$, where ξ_{sh} is *chosen*. Upstream of ξ_{sh}, velocity and temperature profiles will lie close to the u_+ locus, as they will do if $\tilde{\sigma}_0 = \tilde{\sigma}_{0\,\mathrm{crit}}$ in (66) and (67) for example, until a transitional region similar to the one implied by (37) is encountered, which matches with the upstream outskirts of the inert shock wave. Downstream parts of the shock match, via another (67)-type domain, into combustion-dominated regions near the subsonic u_--locus and thence to completion at u_{b-}. This state of affairs will shortly be illustrated by the example of a shock wave within the induction region, but before doing this one must note the following.

If, in the same $o(1)$ category of $\tilde{\sigma}_{0\,\mathrm{crit}} - \tilde{\sigma}_0$ differences that has just been described, one decides *not* to introduce an inert shock into the field, then a continuation of the asymptotic analysis that retains only the linear terms in (66) and (67) will provide an estimate of behaviour in the fully supersonic combustion field exemplified by curve (iv) on figure 3.

Let us calculate the value of the slope of the u_+ locus at $\xi = 0$; (24) already shows that, when $\xi = 0$,

$$u'_+ = -\frac{2T'_{s0}}{(\gamma+1)\,M^2(u_{i+} - u_{i-})},$$

while from (22) one finds that $u_{i+} - u_{i-} = 2u_{i+} - (f - \tilde{\sigma})$. In the particular conditions that apply to this section $u_{i+} \approx 1$ and $\tilde{\sigma}$ is $o(1)$; thus

$$u_{i+} - u_{i-} \approx 2 - f = 2(1 - M^{-2})/(\gamma+1),$$

where the last result follows from (12). It follows that

$$u'_+ = -\frac{T'_{s0}}{M^2-1} \approx \frac{-R_0}{M^2-1} = \frac{-g_i}{M^2-1},$$

where the second and third results derive from (1) (since $T''_s \ll T'_s$, see §2) and (58). Thus u'_+ is equal to $\tilde{\sigma}_{0\,\mathrm{crit}}$, as can be seen from (58) and (75), and it can be seen with the aid of (D 3) that the criterion (75) is just what one needs to ensure that a solution curve on figure 3 starts (at $u = 1$, $\xi = 0$) just below the u_{0+} locus defined in §5. This result gives additional weight to the link between solution curves (iv) and (v) on figure 3 and the present asymptotic analysis, as well as that of §11.

11. A shock within the induction domain

When $\tilde{\sigma}_0$ is arbitrarily close to $\tilde{\sigma}_{0\,\mathrm{crit}}$, defined in (75), $|u-1|$ and $|T-1|$ both increase linearly with ξ (cf. (59), (66) and (67)). Since $M > 1$, $\phi > 0$, and $u-1$ diminishes while $T-1$ increases with ξ. When ξ is $O(1/\theta g_\mathrm{i})$ the validity criterion (76) is violated and new asymptotic estimates are required. It is not difficult to see that these must be

$$T \sim 1+\frac{1}{\theta}T(\Xi), \tag{78}$$

$$u \sim 1+\frac{1}{\theta}U(\Xi),$$

where

$$\Xi = \theta g_\mathrm{i}\,\xi \tag{79}$$

(N.B. g_i is defined in (58)). Substituting (78) and (79) into (1), (2) and (3), and using the limit $\theta \to \infty$ with Ξ fixed, leads in the usual quite straightforward way to the solutions

$$T = -\ln\left(1-\frac{\Xi}{\alpha}\right), \quad \alpha \equiv \frac{M^2-1}{\gamma M^2-1} > 0, \tag{80), (81}$$

$$U = \frac{1}{\gamma M^2-1}\ln\left(1-\frac{\Xi}{\alpha}\right), \tag{82}$$

which describe behaviour in an induction domain (e.g. Kassoy 1976). Since the principal physical influences in such a region are convection and reaction it is called a CR region. The influence of the, in this case supersonic, gasdynamical effects is incorporated in the constant α and we observe the estimate, via (80), that thermal runaway or ignition will occur as $\Xi \to \alpha$ from below.

But now let us suppose that the σ_0 (or $\tilde{\sigma}_0$) value is such as to lead to a CD shock wave *within* this CR induction domain and therefore ahead of the runaway location near $\Xi = \alpha$.

Reverting to ξ-type coordinates for the moment, assume that the interior of the shock is located at $\xi = \xi_\mathrm{sh}$, where, noting (22),

$$u = \tfrac{1}{2}(u_{\mathrm{sh}+}+u_{\mathrm{sh}-}) = \tfrac{1}{2}(f-\hat{\sigma}) \approx \tfrac{1}{2}f, \tag{83}$$

and, also from (22),

$$T_\mathrm{s}(\xi_\mathrm{sh}) \equiv T_{\mathrm{s\,sh}} = \tfrac{1}{2}(\gamma+1)\,M^2\{f-1+\hat{F}(\xi_\mathrm{sh})\} = \tfrac{1}{2}(\gamma+1)\,M^2 u_{\mathrm{sh}+}u_{\mathrm{sh}-}. \tag{84}$$

Thus, around ξ_sh, u is given by

$$u \sim u_\mathrm{sh}(\xi-\xi_\mathrm{sh}), \tag{85}$$

where u_sh satisfies the CD relation

$$\Gamma u_\mathrm{sh}u'_\mathrm{sh} = (u_\mathrm{sh}-u_{\mathrm{sh}+})\,(u_\mathrm{sh}-u_{\mathrm{sh}-}). \tag{86}$$

The solution of (86) that satisfies (83) is

$$u_{\mathrm{sh}+}\ln\frac{u_{\mathrm{sh}+}-u_\mathrm{sh}}{\frac{1}{2}(u_{\mathrm{sh}+}-u_{\mathrm{sh}-})} - u_{\mathrm{sh}-}\ln\frac{u_\mathrm{sh}-u_{\mathrm{sh}-}}{\frac{1}{2}(u_{\mathrm{sh}+}-u_{\mathrm{sh}-})} = \frac{u_{\mathrm{sh}+}-u_{\mathrm{sh}-}}{\Gamma}(\xi-\xi_\mathrm{sh}). \tag{87}$$

As $u_\mathrm{sh} \to u_{\mathrm{sh}+}$, (86) shows that u'_sh begins to diminish like $u_\mathrm{sh}-u_{\mathrm{sh}+}$, as does the corresponding T' value, since T_s is equal to $T_{\mathrm{s\,sh}}$ in this vicinity (see (84)). Hence there will be a CDR region, around ξ_T, say, within which it is no longer correct to ignore the existence of R and the local variations of T_s, as is implicit in (85) and (86).

Defining

$$u \sim u_{sh+} + g_{sh+} U_T(\xi - \xi_T), \quad U_T(0) = 1, \tag{88}$$

$$T \sim T_{sh+} + g_{sh+} \tau_T(\xi - \xi_T),$$

it can readily be shown that, if

$$g_{sh+} = A_4 M^{-2} \theta^s \exp\left(\frac{-\theta}{T_{sh+}}\right)(T_{sb} - T_{s\,sh}) \tag{89}$$

and if and only if

$$\theta g_{sh+} \tau_T / T_{sh+}^2 = o(1), \tag{90}$$

then the functions τ_T and U_T obey equations like (39), (40) and (42), with the following changes; for τ_{si} read τ_{sT}, where

$$\tau_{sT} = \tau_T + (\gamma - 1) M^2 u_{sh+} U_T, \tag{91}$$

for u_{i-} read u_{sh+}, and for U_i read U_T. By similar reasoning to that given between (45) and (46), one finds that

$$\tau_{sT} = \hat{C}_T + \xi - \xi_T, \tag{92}$$

where $\hat{C}_T$ is a constant to be found from matching. Defining B via

$$[2u_{sh+} - (f - \hat{\sigma})] = (u_{sh+} - u_{sh-}) = \Gamma u_{sh+} B > 0, \tag{93}$$

where (83) has been used here, it can be shown that the solution for U_T is exactly the same as (49) for U_i, with the following changes: redefine a in (48) so that u_{sh+} replaces u_{i-}, replace A by $-B$ and replace $\hat{C}$ by $\hat{C}_T$.

It is now necessary to match this CDR solution for U_T with the CD solution (87) *and* with the CR induction solution (82). The reason for the need to match with both (87) and (82) is that there are essentially three unknown quantities in the solutions at this juncture, namely $\hat{C}_T$, ξ_T *and* $T_{s\,sh}$. Although one can select ξ_{sh} at will, it is still necessary to solve the system of equations (1)–(3) in order to *find* $T_{s\,sh}$; this is being done here by asymptotic means. Matching of the U_T solution with (87) proceeds in a way closely akin to that described between (49) and (51) in §8. It transpires that

$$B(\xi_{sh} - \xi_T) = \frac{\theta}{T_{sh+}} - \ln\{A_4 M^{-2} \theta^s (T_{sb} - T_{s\,sh})\}, \tag{94}$$

$$\hat{C}_T \frac{\hat{a}}{B} + \frac{\hat{a}}{B^2} + 1 = -\tfrac{1}{2}(u_{sh+} - u_{sh-})\, 2^{u_{sh-}/u_{sh+}}, \tag{95}$$

where, as described below (93), $\hat{a}$ is a in (48) with u_{sh+} in place of u_{i-}:

$$\hat{a} = 1/\gamma M^2 u_{sh+}. \tag{96}$$

In order to complete the matching with (78) and (82) it is helpful to define an intermediate variable Ξ_{in} as follows. First rewrite Ξ from (79) as

$$\Xi = \theta g_i \xi = \theta g_i \tilde{\xi} + \theta g_i \xi_T, \quad \tilde{\xi} = \xi - \xi_T. \tag{97}$$

Since the shock is hypothesized to be within the induction domain, $\Xi < \alpha$, and in view of (94), one can define

$$\alpha_T \equiv \theta g_i \xi_T \tag{98}$$

and be certain that $\alpha_T < \alpha$. Then one defines

$$\Xi_{in} = g_{in} \tilde{\xi}, \tag{99}$$

where $g_i/g_{in} = o(1)$, and notes that (97)–(99) make

$$\Xi = \alpha_T + \frac{g_i}{g_{in}} \Xi_{in}. \tag{100}$$

Writing the induction solution in intermediate coordinates and using $\theta \to \infty$ with Ξ_{in} fixed shows that

$$u \sim 1 + \frac{1}{(\gamma M^2 - 1)\theta}\left\{\ln\left(1 - \frac{\alpha_T}{\alpha}\right) - \frac{g_i \Xi_{in}}{g_{in}(\alpha - \alpha_T)} + \dots\right\}. \tag{101}$$

Doing the same thing for the U_T solution gives

$$u \sim u_{sh+} + g_{sh+}\left\{-\frac{\hat{a}}{B}\frac{\Xi_{in}}{g_{in}} + o\left(\frac{1}{g_{in}}\right)\right\}, \tag{102}$$

and matching between (101) and (102) is effected if one can reconcile

$$u_{sh+} \quad \text{with} \quad 1 + \frac{1}{(\gamma M^2 - 1)\theta}\ln\left(1 - \frac{\alpha_T}{\alpha}\right) \tag{103}$$

and

$$-\frac{g_{sh+}}{g_{in}}\frac{\hat{a}}{B}\Xi_{in} \quad \text{with} \quad -\frac{1}{(\gamma M^2 - 1)\theta}\frac{g_i}{g_{in}}\frac{\Xi_{in}}{(\alpha - \alpha_T)}. \tag{104}$$

Relation (104) is of central importance; using the definitions of g_i and g_{sh+} from (58) and (89) respectively, (104) requires

$$(T_{sb} - T_{s\,sh})\frac{\hat{a}}{B}\exp\left(\frac{-\theta}{T_{sh+}}\right) = (T_{sb} - T_{so})\frac{e^{-\theta}}{\theta(\gamma M^2 - 1)(\alpha - \alpha_T)}. \tag{105}$$

Thus T_{sh+} must be like $1 + O(\theta^{-1})$ in order for the exponentials to be of the same order. Define φ_{sh} via

$$T_{sh+} = 1 + \frac{1}{\theta}\varphi_{sh}, \tag{106}$$

and note that

$$T_{s\,sh} = T_{sh+} + \tfrac{1}{2}(\gamma - 1) M^2 u_{sh+}^2.$$

Since both T_{sh+} and u_{sh+} (see requirement (103)) must be of the form $1 + O(\theta^{-1})$ it follows that

$$T_{s\,sh} = T_{so} + O(\theta^{-1}), \tag{107}$$

where T_{so} is defined in (5). Then (106) and (107) in (105) show that, to first order, φ_{sh} is such that

$$\frac{\hat{a}}{B} e^{\varphi_{sh}} = (\gamma M^2 - 1)^{-1}(\alpha - \alpha_T)^{-1}.$$

But it can be shown that

$$\frac{\hat{a}}{B} = \frac{1}{M^2 - 1} = \frac{1}{\alpha(\gamma M^2 - 1)}$$

(use (93) and (96) and the various relations between $u_{sh\pm}$, f etc., as well as (81) for α) so that

$$\varphi_{sh} = -\ln\left(1 - \frac{\alpha_T}{\alpha}\right),$$

$$T_{sh+} = 1 - \frac{1}{\theta}\ln\left(1 - \frac{\alpha_T}{\alpha}\right). \tag{108}$$

Together with

$$u_{sh+} = 1+\frac{1}{\theta(\gamma M^2-1)}\ln\left(1-\frac{\alpha_T}{\alpha}\right), \tag{109}$$

which is just a restatement of (103), the relation (108) completes the matching, and hence the upstream structure sequence of a CR induction domain, a CDR transitional domain (specifically (88) etc.) and a CD shock.

The matching of these various domains has been described here in some little detail in order to emphasize the mathematical and physical unity and solidity of the asymptotic results, especially in the context of the general predictions of curve-(v)-type solutions on figure 3, described in §5.

Evidently the downstream link between a shock and subsequent subsonic events near the u_- locus will be very like the one described in §8, but displaced from $\xi = 0$ to the selected ξ_{sh} location. These matters will not be discussed any further at this juncture, save to remark that the gasdynamical jump of temperature at the shock will have a dramatic effect on the rate of progress of the downstream chemical activity.

It should be noted from (117) that ξ_{sh} differs from ξ_T by an amount that is $O(\theta)$ at most. Defining

$$\alpha_{sh} \equiv \theta g_i \xi_{sh}, \tag{110}$$

it can be seen from (94) and (98) that

$$\alpha_{sh} = \alpha_T + O(\theta^2 g_i), \tag{111}$$

where g_i is given in (58). Thus α_{sh} and α_T differ by only exponentially small terms in the θ-limit, and one can replace α_T, in (108) and (109) for example, by α_{sh} without affecting the validity of the results to first order. There is in fact some merit in doing this since it makes the results for T_{sh+} and u_{sh+} fit more neatly with the hypothesis that the shock is centred at ξ_{sh}.

Having now determined T_{sh+} and u_{sh+} up to and including $O(1/\theta)$ it is a simple matter to find $T_{s\,sh}$ to the same order of accuracy and then, through (83) in particular, to find u_{sh-} and T_{sh-}.

The CD domain behaviour is described to first order by (87) as $T \to T_{sh-}$ and $u \to u_{sh-}$, at least until the R-term in (1) becomes comparable to the convection and diffusion effects. This downstream CDR type of domain will be described in the usual way by using the asymptotic representations

$$u \sim u_{sh-} + g_{sh-} U_D(\xi-\xi_D), \quad U_D(0) = 1, \tag{112}$$

$$T \sim T_{sh-} + g_{sh-} \tau_D(\xi-\xi_D). \tag{113}$$

In present circumstances

$$g_{sh-} = A_4 M^{-2} \theta^s \exp\left(\frac{-\theta}{T_{sh-}}\right)(T_{sb}-T_{s\,sh}). \tag{114}$$

Provided that

$$\theta g_{sh-} \tau_D / T^2_{sh-} = o(1), \tag{115}$$

the solutions for U_D and τ_D provide valid asymptotic estimates (112) and (113) for u and T. The actual solutions for τ_D and U_D are very like those for τ_T and U_T, or τ_i and U_i (see §8).

Writing

$$\tau_{sD} = \tau_D + (\gamma-1) M^2 U_D u_{sh-}, \tag{116}$$

it can be shown that

$$\tau_{sD} = \hat{C}_D + (\xi-\xi_D), \tag{117}$$

where ξ_D and $\hat{C}_D$ are found from matching conditions to be as follows:

$$\bar{A}(\xi_D - \xi_{sh}) = \frac{\theta}{T_{sh-}} - \ln\left[A_4 M^{-2}\theta^s (T_{sD} - T_{ssh})\right], \tag{118}$$

$$\frac{\tilde{a}}{\bar{A}}\hat{C}_D + \frac{\tilde{a}}{\bar{A}^2} + 1 = \tfrac{1}{2}(u_{sh+} - u_{sh-})\, 2^{u_{sh+}/u_{sh-}}, \tag{119}$$

$$\bar{A} = \frac{u_{sh+} - u_{sh-}}{\Gamma u_{sh-}}, \quad \tilde{a} = \frac{1}{\gamma M^2 u_{sh-}}. \tag{120}$$

The solution for U_D is exactly the same as the result for U_i in (49), with the following changes: $\bar{A}$ is written for A, $\tilde{a}$ for a, $\hat{C}_D$ for $\hat{C}$ and ξ_D for ξ_i.

The CDR-domain solution is now formally complete. One can see from (116), (117) and the solution for U_D that τ_D is changing linearly with $\xi - \xi_D$ when this latter quantity is sufficiently large and positive, as of course is U_D itself. In view of the positive character of $\tilde{a}$ and $\bar{A}$ there is no doubt that U_D is increasing with $\xi - \xi_D$, but the behaviour of τ_D is not quite so obvious. Note first from (116) and (117) that

$$\tau'_{sD} = \tau'_D + (\gamma - 1) M^2 u_{sh-} U'_D = 1,$$

so that as $\xi - \xi \to \infty$

$$\tau'_D \approx 1 - (\gamma - 1) M^2 u_{sh-} \frac{\tilde{a}}{\bar{A}}$$

because $U'_D \approx \tilde{a}/\bar{A}$ under these conditions. Using the definitions of $\tilde{a}$ and $\bar{A}$ from (120), it follows that

$$\tau'_D \approx 1 - 2\frac{\gamma - 1}{\gamma + 1}\frac{u_{sh-}}{u_{sh+} - u_{sh-}}. \tag{121}$$

Equation (109) shows that u_{sh+} is less than unity by an amount that is $O(1/\theta)$; then (83) shows that u_{sh-} is equal to $f - 1$ plus the same $O(1/\theta)$ quantity. To a first order of approximation that neglects these $O(1/\theta)$ terms, (121) can be rewritten as

$$\tau'_D \approx 1 - \frac{2(\gamma - 1)(f - 1)}{(\gamma + 1)(2 - f)}. \tag{122}$$

Thus τ'_D is only positive if M^2 exceeds the value $(3\gamma - 1)/\gamma(3 - \gamma)$, as can be seen by using (12). These particular values of the Mach number of the shock wave are just what are required to make the Mach number of the flow downstream less than or equal to the value $\gamma^{-\frac{1}{2}}$ or, in other words, to make the downstream flow subsonic relative to the local *isothermal* sound speed. The situation is just the same as the one described in the paragraph that follows (69); the diminishing temperature will slow down the rate of combustion-energy release and so prevent the onset of ignition which, before the advent of the shock, appears (from (78)–(80)) to be inevitable when $\Xi \to \alpha$.

When $1 < M^2 < (3\gamma - 1)/\gamma(3 - \gamma)$ the dynamics of gaseous behaviour act to prevent ignition, and combustion proceeds, so far as the present analysis indicates, as a relatively slow decomposition of the fuel species. When $M > 1$ and the shock lies inside the induction zone, as has been postulated throughout this section, the M-value to ensure isothermally subsonic flow behind the shock must be greater than $(3\gamma - 1)/\gamma(3 - \gamma)$ by an amount that is $O(1/\theta)$.

The linear dependence of τ_D on $\xi - \xi_D$ will eventually lead to a failure of the condition (115). This signifies the appearance of a new domain of the CR type, within which

$$T \sim T_{sh-} + \frac{1}{\theta} T_D(\Xi_D), \quad (123)$$

$$u \sim u_{sh-} + \frac{1}{\theta} U_D(\Xi_D),$$

and the new coordinate is

$$\Xi_D = \theta g_{sh-}(\xi - \xi_D)/T_{sh-}^2, \quad (124)$$

where g_{sh-} is defined in (114); ξ_D is given in (118). Calculation of T_D and U_D, including matching of the solutions with (112) and (113), is straightforward and there is nothing to be gained here by describing all, or any, of the intermediate steps. It is advantageous to use the local Mach number m_{sh-}, defined by

$$m_{sh-}^2 = M^2 u_{sh-}^2 / T_{sh-}, \quad (125)$$

in displaying the results, and it is noted in passing that τ_D' in (121) can be re-expressed in these terms by the relation

$$\tau_D' \approx \frac{1-\gamma m_{sh-}^2}{1-m_{sh-}^2} \equiv \frac{1}{\alpha_{sh-}}. \quad (126)$$

Note the results for τ_i that follow from (67) and (68), and compare them with (126); also note the definition of α in (81), and compare with (126). It is found that

$$T_D = -T_{sh-}^2 \ln\left\{1 - \frac{\Xi_D}{\alpha_{sh-}}\right\}, \quad (127)$$

$$\frac{U_D}{u_{sh-}} = \frac{T_D}{T_{sh-}(1-\gamma m_{sh-}^2)}, \quad (128)$$

and these results should be compared with (80)–(83).

Evidently there is no dramatic change in the rough general character of the events that the shock wave has interrupted in the neighbourhood of ξ_{sh}, but one must take heed of two important modifications that it *has* brought about. The first is the substantial shortening of the lengthscale of the combustion activity, since $g_{sh-} \gg g_i$, as can be seen from (58) and (114) and the fact that $T_{sh-} > 1$. This is an illustration of the intuitively obvious effect of shock heating; the illustration here is reinforced by specific formulae and can therefore be quantified if desired.

The second point to be made is in many ways a warning about 'intuitively obvious' effects, and enlarges somewhat on the remarks made below (122). If the shock is too weak, so that it drives the flow to a local speed that is (essentially) subsonic relative to the isentropic frozen sound speed but *supersonic* relative to the local isothermal or Newtonian frozen sound speed, then even though the local rate of chemical activity is speeded up behind the shock compressibility effects supervene and local temperatures actually decrease below T_{sh-}. This can be seen from (127), which shows that when $\alpha_{sh-} < 0$ ($\gamma m_{sh-}^2 > 1$, $m_{sh-}^2 < 1$) T_D is always negative and diminishing as Ξ_D increases. Equation (128) shows that U_D, and hence u, is increasing; thus chemical energy is increasing flow kinetic energy at the expense of thermal energy. This gasdynamical suppression of ignition will be examined in more detail in the second part of this work.

When $\alpha_{sh-} > 0$ ($\gamma m_{sh-}^2 < 1$) an estimate of the onset of ignition is found from (127), namely

$$\Xi_D \to \alpha_{sh-}. \quad (129)$$

Subsequent events will be described in another paper. Observe that the asymptotic analysis provides us with an estimate of the location of these events relative to $\xi = 0$ via equations (124) and (118) for any chosen ξ_{sh}; in the present section ξ_{sh} must lie within the induction zone, by hypothesis, and it has already been remarked that this requires $\xi_{sh} < \alpha/\theta g_i$ (see (79)–(81)).

12. Conclusions

The reader is reminded that M is the Mach number of the flow at inlet to the system through the face of the flameholder, while m is a general local value of the Mach number; σ_0 is proportional to the gas velocity gradient at inlet and is therefore influenced by the allowable changes of pressure across the field of flow. Some of the main results of the present work are as follows.

1. For the given model of a reacting compressible flow a unique continuous field structure exists for all M within the range of allowable σ_0 values.

2. For $M < 1$ almost inert zones of gasdynamical adjustment exist at the face of the flameholder; they act to reduce local velocity gradients from σ_0 to values that are then compatible with a balance between convection and reaction alone, in which mode most of the combustion energy is released.

3. For $M > 1$ the remarks in item 2 apply for most allowable σ_0 values, which give rise to shock-wave-like zones of almost-inert gasdynamical behaviour at the face of the flameholder. However, there is a very small critical negative value of σ_0 which, when σ_0 is near to and below it, gives rise to shock waves that lie well within the combustion activity and far from the flameholder.

4. For σ_0 within a very close neighbourhood of its critical value a supersonic (i.e. $m > 1$ everywhere) combustion field exists.

5. Inlet M-values lying between the classical upper and lower Chapman–Jouguet values are permitted; however, a gasdynamical adjustment zone then always exists at the flameholder which reduces m to a value less than or equal to the lower CJ value before significant release of combustion energy begins.

6. When $M < 1$ it must also be less than $\gamma^{-\frac{1}{2}}$, otherwise ignition is not possible; the kinetic energy of the flow increases at the expense of thermal energy when $1 > M > \gamma^{-\frac{1}{2}}$. When $M > 1$ it must be large enough to make m behind the shock (wherever the shock may be within the induction zone) less than $\gamma^{-\frac{1}{2}}$, for the reasons just given.

The asymptotic analysis shows throughout that the field consists of thin, effectively inert, zones of convection–diffusion balance (gasdynamical adjustment) joined, by somewhat thicker transition layers, in which convection, diffusion and reaction all play a part, to long domains in which convection–reaction balances dominate and in which the major part of the combustion energy is released. There is no *a priori* reason why a porous-plug burner could not be run with supersonic speeds at the outlet; although it may prove to be quite difficult to manufacture a sufficiently 'small-grain' porosity with the necessary internal convergent–divergent duct structure it may be worth attempting the task so that studies of combustion in simple compressible-flow configurations could be undertaken.

The above is an account of work carried out between 25 May and 3 August 1982 while the writer was a guest of Professor D. R. Kassoy and Professor J. Bebernes at the University of Colorado, Boulder. It is a pleasure to be able to record my thanks to my hosts for their invitation and for the stimulus of their company during this period. Thanks are also due to the U.S. Army Research Office, who made the visit possible by their generous provision of financial support under contract DAAG29-82-K-0069.

Appendix A. Limitations on $\hat{\sigma}$

From (28), potentially admissible values for u_b are given by

$$u_{b\pm} = \tfrac{1}{2}(f-\hat{\sigma}) \pm |\{\tfrac{1}{4}(f-\hat{\sigma})^2 - (f-1+F)\}^{\frac{1}{2}}|, \tag{A 1}$$

and it is at once clear that $\hat{\sigma}$ cannot be chosen with complete freedom for any given f and F (or M and $\hat{Q}$) values. In order for the last part of (A 1) to be real one must find

$$(f-2)^2 - 4F \geqslant (2f-\hat{\sigma})\,\hat{\sigma}.$$

In other words, one must find $\hat{\sigma} \geqslant \hat{\sigma}_+$, or $\hat{\sigma} \leqslant \hat{\sigma}_-$, where

$$\hat{\sigma}_\pm = f \pm 2|(f-1+F)^{\frac{1}{2}}|. \tag{A 2}$$

Since $f-1 > 0$ and $F > 0$ (because $\tilde{Q} \geqslant 0$) it follows that $\hat{\sigma}_\pm$ are both always real, with $\hat{\sigma}_+ > f$. The latter implies that, when $\hat{\sigma} = \hat{\sigma}_+$,

$$u_{b+} = u_{b-} = \tfrac{1}{2}(f-\hat{\sigma}_+) < 0,$$

as can be seen from (A 1). This last equation makes it clear that $u_{b\pm} < 0$ for all $\hat{\sigma} \geqslant \hat{\sigma}_+$, since $f-1+F > 0$, and so these conditions on $\hat{\sigma}$ are of no physical significance.

A general limitation on the value of $\hat{\sigma}$ is therefore

$$\hat{\sigma} \leqslant \hat{\sigma}_- \equiv f - 2|(f-1+F)^{\frac{1}{2}}|. \tag{A 3}$$

Note that $\hat{\sigma}_- \leqslant 0$ if

$$4F \geqslant (f-2)^2. \tag{A 4}$$

This condition can be translated into terms of M, γ and $\tilde{Q}$ through the relations in (13) and (14) and reads

$$(1-M^2)^2 \leqslant 2(\gamma+1)\,\tilde{Q}M^2. \tag{A 5}$$

The equality sign in (A 5) defines a special pair of inlet Mach numbers, namely the Chapman–Jouguet Mach numbers,

$$M^2_{CJ\pm} = \{1+(\gamma+1)\,\tilde{Q}\} \pm |[\{1+(\gamma+1)\,\tilde{Q}\}^2 - 1]^{\frac{1}{2}}| \gtrless 1, \tag{A 6}$$

so that

$$M_{CJ-} < M < M_{CJ+} \Leftrightarrow \hat{\sigma} < \hat{\sigma}_- < 0. \tag{A 7}$$

Thus the inlet Mach number in the present configuration *can* lie in the domain of Mach numbers that is described as forbidden or inaccessible in the classical Hugoniot-curve and Rayleigh-line analysis of metastable oncoming streams, *provided* that $\hat{\sigma} < \hat{\sigma}_- < 0$, where $\hat{\sigma}_-$ is defined in (A 3). The further implications of this result are worked out in §7.

Equation (12) shows that when

$$0 < M \leqslant 1, \quad f \geqslant 2 \tag{A 8}$$

while when

$$\infty > M \geqslant 1, \quad 1 < \Gamma < f \leqslant 2. \tag{A 9}$$

These facts are helpful in the interpretation of the locus of possible end-states as a function of $\hat{\sigma}$ for any given values of M and γ (and hence f and Γ).

Appendix B. Flow pressure and its relation to $\hat{\sigma}$

The present model assumes that the dimensional pressure $\bar{p}$, density $\bar{\rho}$ and temperature $\bar{T}$ are simply related by

$$\bar{p} = \bar{\rho}R_{\mathrm{w}}\bar{T} = \bar{\rho}\bar{u}R_{\mathrm{w}}\frac{\bar{T}}{\bar{u}} = \bar{m}R_{\mathrm{w}}\frac{\bar{T}}{\bar{u}},$$

where R_{w} is a fixed value of the gas constant and $\bar{m}$ is the constant mass flux per unit area. At the flameholder

$$\bar{p}_0 = \bar{m}R_{\mathrm{w}}\frac{\bar{T}_0}{\bar{u}_0},$$

so that, defining the dimensionless absolute pressure to be

$$p = \bar{p}/\bar{p}_0, \tag{B 1}$$

it follows that

$$p = \frac{T}{u} = \frac{T_{\mathrm{s}}}{u} - \tfrac{1}{2}(\gamma-1)\,M^2 u. \tag{B 2}$$

The last result follows from (3); using (22) we can write

$$p = \tfrac{1}{2}M^2\left\{(\gamma+1)\frac{u_+u_-}{u} - (\gamma-1)\,u\right\} \tag{B 3}$$

in general. In the particular circumstances for which

$$u = u_{\pm},$$

the corresponding values of p, namely $p_{\pm}$, are

$$p_{\pm} = \tfrac{1}{2}M^2\{(u_+ + u_-) \mp \gamma(u_+ - u_-)\}. \tag{B 4}$$

It can be shown that $\gamma(u_+ - u_-) \ngtr (u_+ + u_-)$, so that p_+ is always positive, as, of course, is p_- *a fortiori*.

Since $u_+ + u_-$ is constant (see (22)) while $u_+ - u_-$ diminishes as ξ increases, it follows that p_+ increases while p_- decreases in these circumstances.

It is also rather easy to see how the final pressures $p_{\mathrm{b}\pm}$ depend upon parameters like M and σ_0 by reverting to (2). Making use of (B 2), it follows that

$$p_{\mathrm{b}\pm} = 1 + \gamma\, M^2\{1 - u_{\mathrm{b}\pm} - \sigma_0\},$$

since $u' \to 0$ as $u \to u_{\mathrm{b}\pm}$. The values $u_{\mathrm{b}\pm}$ are given in (A 1).

Appendix C. Subsonic and supersonic flow domains

Note first that the local Mach number in the flow is given by

$$m^2 = M^2u^2T^{-1}. \tag{C 1}$$

Define

$$m_{\pm}^2 = M^2u_{\pm}^2\,T_{\pm}^{-1} \tag{C 2}$$

so that

$$T_{\mathrm{s}} = T_{\pm} + \tfrac{1}{2}(\gamma-1)\,M^2u_{\pm}^2 = M^2u_{\pm}^2\{m_{\pm}^{-2} + \tfrac{1}{2}(\gamma-1)\}. \tag{C 3}$$

Since (22) gives

$$u_+u_- = \frac{2T_{\mathrm{s}}}{(\gamma+1)\,M^2}, \tag{C 4}$$

it follows from (C 3) and (C 4) that

$$(m_+^{-2}+\tfrac{1}{2}(\gamma-1)1)(m_-^{-2}+\tfrac{1}{2}(\gamma-1)) = \tfrac{1}{4}(\gamma+1)^2, \qquad \text{(C 5)}$$

from which it can be deduced that

$$m_+ > 1, \quad m_- < 1. \qquad \text{(C 6)}$$

Recall that $u_+ > u_- \Rightarrow T_+ < T_-$ at the same stagnation temperature. Thus u_+ is a 'supersonic' curve, while u_- is 'subsonic'. In particular, any solution that ends at u_{b+} describes a flow that emerges from the chemical and gasdynamical interactions as a supersonic stream. Using (C 1), one can write

$$T_s = M^2u^2\{m^{-2}+\tfrac{1}{2}(\gamma-1)\},$$

so that, when $m = 1$ and the flow is locally sonic, one finds that

$$\frac{2T_s}{(\gamma+1)M^2} = u_*^2 = u_+u_-. \qquad \text{(C 7)}$$

The last result follows from (22) and u_* is the sonic speed. The location of the latter is displayed on figures 2 and 3 (the line lies below $\frac{1}{2}(u_+ + u_-)$ because this sum is always greater than $(u_+u_-)^{\frac{1}{2}}$ for any $u_+ \neq u_-$). It can now be seen that any solution curve on figure 3 that stays above the sonic line u_*, having begun at $u = 1$ with $M > 1$, represents a case of fully supersonic combustion.

Appendix D. The loci $u_{0\pm}$

In order to investigate the small-$\hat{\sigma}$ case, and indeed to make more precise the meaning of the word small in this context, it is necessary to acquire some more information about integral-curve behaviour for (19) when these curves are initially in close proximity to the supersonic u_+ locus. It is important to know whether integral curves that lie between the u_+ and u_- loci can ever escape from that region across the u_+ line (the solutions (i) and (ii) sketched on figure 2 demonstrate that escape into $u < u_-$ is not only possible but necessary for the acquisition of an ultimately subsonic solution). To answer this question define

$$U = u_+(\xi)-u, \qquad \text{(D 1)}$$

and examine the sign of U'. If $U' < 0$ the integral curve must, locally, be converging on u_+, whilst when $U' > 0$ the integral curve will diverge from u_+ locally. From (19) and (D 1) it follows that

$$\Gamma U' = \Gamma u_+' + (u_+-u)\left(1-\frac{u_-}{u}\right). \qquad \text{(D 2)}$$

Recalling (24), which shows that $u_+' < 0$ and, incidentally, that it is of small absolute magnitude since T_s' has this behaviour, (D 2) shows that $U' = 0$ for u-values near to both u_+ and u_-. To a good approximation, the values $u_{0\pm}$ of u at which U' vanishes are given by

$$u_{0+} \approx u_+ - \Gamma|u_+'|\left(1-\frac{u_-}{u_+}\right)^{-1}, \qquad \text{(D 3a)}$$

$$u_{0-} \simeq u_- + \Gamma|u_+'|\left(\frac{u_+}{u_-}-1\right)^{-1}. \qquad \text{(D 3b)}$$

If $\hat{\sigma} \rightarrow \hat{\sigma}_-$, (A 1) and (A 3) make it clear that $u_{b+} \rightarrow u_{b-}$; thus $u_+ \rightarrow u_-$ as $\xi \rightarrow \infty$, and it is possible for u_{0+} and u_{0-} to first merge and then cease to exist for some large ξ-values in these particular conditions. This fact has no radical effect on the character of the solutions.

The existence of a locus u_{0-} in the neighbourhood of u_- is confirmation of the sort of solution-curve behaviour depicted in figure 2, curves (i) and (ii), but does not add anything new to our understanding of this behaviour.

Equation (D 3*a*) describes a locus of points at which U' vanishes; it should be observed that it is *not* parallel to u_+ for the twin reasons that $|u'_+|$ and $(1-u_-/u_+)$ vary with ξ. In general terms, $|u'_+|$ will grow to a maximum value as ξ increases from zero and then subside to zero as $\xi \rightarrow \infty$; u_{0+} therefore approaches u_+ as $\xi \rightarrow \infty$. The quantity $(1-u_-/u_+)$ diminishes in size as ξ increases, and so u_{0+} starts close to u_+ at $\xi = 0$, moves away from it as ξ increases, and finally approaches u_+ as $\xi \rightarrow \infty$. It is readily seen from (D 2) that $U' < 0$ for u in $u_{0+} < u \leqslant u_+$ and $u_- < u < u_{0-}$, while $U' > 0$ for $u_{0-} < u < u_{0+}$.

For a fixed value of ξ, and therefore of u_+, u_- and u'_+, (D 2) shows that

$$\frac{\mathrm{d}U'}{\mathrm{d}u} = -1 + \frac{u_+u_-}{u^2}, \quad \frac{\mathrm{d}^2U'}{\mathrm{d}u^2} = -2\frac{u_+u_-}{u^3} < 0.$$

Thus U' takes its maximum value of

$$U'_{\max} = u'_+ + (u_+^{\frac{1}{2}} - u_-^{\frac{1}{2}})^2$$

on the u-curve that crosses the sonic line $u_* = (u_+u_-)^{\frac{1}{2}}$. Note that if $u_+ \rightarrow u_-$ too closely it is possible to find $U'_{\max}$ with the same sign as u'_+, namely negative; this is confirmation of the remarks made after (D 3*b*).

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